

WORLDSHEETS WITH EXTENDED SUPERSYMMETRY**Laurent Baulieu**¹*LPTHE, Universités Paris VI - Paris VII, Paris, France*²

and

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We determine the equations which govern the gauge symmetries of worldsheets with local supersymmetry of arbitrary rank (N, N') , and their possible anomalies. Both classical and ghost conformally invariant multiplets of the left or right sector are assembled into the components of a single $O(N)$ -superfield. The component with ghost number zero of this superfield is the N -supersymmetric generalization of the Beltrami differential. In a Lagrangian approach, and after gauge-fixing, it becomes the super-moduli of Riemann surfaces coupled to local supersymmetry of rank N . It is also the source of all linear superconformal currents derived from ordinary operator product techniques. The interconnection between BRST invariant actions with different values of $N \geq 3$, and their possible link to topological 2D-gravity coupled to topological sigma models, are shown by straightforward algebraic considerations.

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1 Introduction

The quantum theories based on the symmetries of 2D-gravity coupled to local supersymmetry have proven to be extremely powerful to describe physical phenomena.

The chosen rank of the supersymmetry varies from one application to the other. However, very intriguing relations have been shown to exist between the various theories with different values of N . In particular, the existence of a sort of embedding of N -theories into $(N + 1)$ -theories has been shown from several points of view [1, 2, 3]. This seems to privilege the theory with $N = \infty$, while the $N = 3, 4$ theories play a boundary role between the theories with a relativistic matter interpretation, that is with $N = 0, 1, 2$ and the others.

Here we will show that a kind of universality also exists for the mathematical description of worldsheets with extended supersymmetry, with a surprisingly simple algebraic structure. We point out the possibility of generalizing the very old notion of Beltrami parametrization of conformally invariant Riemann surfaces to the case where reparametrization invariance is coupled to local worldsheet supersymmetry of arbitrary rank (N, N') . We find that the full multiplet of conformally invariant 2D-supergravity of rank N in the left sector is made of all possible 2D gauge fields valued in the antisymmetric tensor representations of $O(N)$, with similar properties in the right sector. Moreover the complete BRST equations of this theory, and thus the algebra of its gauge transformations, follow from a universal vanishing curvature condition in superspace. These will be described in sect. 2.

This formulation of the gauge symmetries of 2D-supergravities as a vanishing curvature condition enables us to identify their ghost actions as a BF systems, which needless to say, emphasizes the topological aspects of these theories. We will briefly elaborate on this by using the Batalin-Vilkovisky (BV) framework [4] in sect. 3.

The BRST symmetry of the ghosts that we obtain coincides with that one obtains from the usual operator product expansion (OPE) treatments of linear superconformal algebra (SCA). Since we define the complete set of gauge fields of conformal 2D-supergravity, we can use them as background gauge fields. This gives quite a simple and clear reversed construction of the currents of SCA by use of these background gauge fields as the sources of the currents.

When gauge fixing the symmetries of a worldsheet with (N, N') local supersymmetry, the gauge fields of 2D-supergravity should not be put equal to zero, since this would result in an over gauge-fixing for higher genera. Rather, they can be at most set equal to constant backgrounds over which one should integrate, by taking also into account the remaining modular invariance.

Our presentation has thus also the advantage of defining all (super)-moduli of extended 2D-supergravities in a well-defined framework.

We illustrate some of our results in the cases of $N = 2, 3$ and (large) $N = 4$ supersymmetry.

We give a combinatoric proof that all pure 2D-supergravities with $N \geq 3$ are anomaly free, which would only allow their couplings to topological matter, in contrast with the cases $N = 2, 1, 0$ for which critical matter must be introduced to compensate for the anomaly. We also elaborate on the question of the consistent anomaly, for which we give explicit expressions for $N = 2, 3$, which generalize the known results for $N = 0, 1$.

Finally, we demonstrate the embedding properties between N - and $(N + 1)$ -theories, by showing that for $N \geq 3$ the BRST invariant action of the former can be considered as part of the latter. We also suggest a link between these theories and topological 2D-gravity coupled to topological sigma models with bosonic and fermionic coordinates.

2 Conformally invariant parametrization for 2D-supergravities

The Beltrami parametrization consists in expressing the squared length elements of the world-sheet as

$$d\sigma^2 = \exp \varphi (dz + \mu_z^{\bar{z}} d\bar{z})(d\bar{z} + \mu_{\bar{z}}^z dz). \quad (1)$$

This parametrization is very natural because the Beltrami differential $\mu_z^{\bar{z}}$ undergoes the reparametrization symmetry under a factorized form, and is dilatation invariant. More precisely, the BRST symmetry transformation of $\mu_z^{\bar{z}}(z, \bar{z})$ only involves a ghost $c^z(z, \bar{z})$ (which can be identified as a suitable combination of the two components of the ordinary reparametrization ghost field). In ref. [5], the extension of the Beltrami parametrization to the case of $N = 1$ 2D-supergravity was found, by introducing the conformally invariant part of the gravitino $\alpha_{\bar{z}}^{\frac{1}{2}}(z, \bar{z})$, with ghost $\gamma^{\frac{1}{2}}(z, \bar{z})$, and the following factorized BRST algebra

$$\begin{aligned} s\mu_z^{\bar{z}} &= \partial_{\bar{z}}c^z + c^z\partial_z\mu_z^{\bar{z}} - \mu_z^{\bar{z}}\partial_zc^z - \frac{1}{2}\alpha_{\bar{z}}^{\frac{1}{2}}\gamma^{\frac{1}{2}}, \\ s\alpha_{\bar{z}}^{\frac{1}{2}} &= -\partial_{\bar{z}}\gamma^{\frac{1}{2}} - \frac{1}{2}\gamma^{\frac{1}{2}}\partial_z\mu_z^{\bar{z}} + \mu_z^{\bar{z}}\partial_z\gamma^{\frac{1}{2}} + c^z\partial_z\alpha_{\bar{z}}^{\frac{1}{2}} + \frac{1}{2}\alpha_{\bar{z}}^{\frac{1}{2}}\partial_zc^z, \\ sc^z &= c^z\partial_zc^z - \frac{1}{4}\gamma^{\frac{1}{2}}\gamma^{\frac{1}{2}}, \\ s\gamma^{\frac{1}{2}} &= c^z\partial_z\gamma^{\frac{1}{2}} - \frac{1}{2}\gamma^{\frac{1}{2}}\partial_zc^z, \end{aligned} \quad (2)$$

This algebra has remarkable algebraic properties, and we will show how it can be generalized to the case of extended supersymmetry of any given rank N . Eq. (2) has a simple superfield

formulation. Indeed, by introducing one single Grassmann variable θ , these BRST equations can be written as

$$\hat{d}\hat{M}^z - \hat{M}^z \partial_z \hat{M}^z + \frac{1}{4}(D_\theta \hat{M}^z)^2 = 0, \quad (3)$$

where $D_\theta = \partial_\theta + \theta \partial_z$ and the BRST transformation operator s and the differential $d = dz \partial_z + d\bar{z} \partial_{\bar{z}}$ are unified into $\hat{d} = d + s$ while the holomorphic part of the conformally invariant classical gauge fields of $N = 1$ 2D-supergravity and of their ghosts are assembled into the superfield $\hat{M}(z, \bar{z}, \theta)$ as $\hat{M}^z = M^z + C^z$ with

$$M^z(z, \bar{z}, \theta) = dz + \mu_{\bar{z}}^z(z, \bar{z}) d\bar{z} + \theta \alpha_{\bar{z}}^{\frac{1}{2}}(z, \bar{z}) d\bar{z}, \quad C^z(z, \bar{z}, \theta) = c^z(z, \bar{z}) + \theta \gamma^{\frac{1}{2}}(z, \bar{z}). \quad (4)$$

By expanding (3) in ghost number and powers of θ one recovers (2). Notice that the property $\hat{d}^2 = s^2 = 0$ implies the $N = 1$ supersymmetry relation $D_\theta^2 = \partial_z$.

Eq. (3) can be understood as a direct consequence of the vanishing of the torsion in superspace, without any reference to the OPE techniques. Since the Beltrami differential and the gravitino are the source of the energy-momentum tensor and of the supersymmetry current, this can be seen as the possibility for a reversed, and perhaps more geometrical, construction of superconformal quantum field theory.

We now generalize this to higher rank supersymmetry. We use the natural framework for extended 2D-supergravity, which, for the holomorphic sector, is the N -superspace with coordinates (z, θ^i) , $i = 1, \dots, N$ [6, 7].

We define the generalization in N -superspace of the one-form which unifies the Beltrami differential and its ghost as

$$\begin{aligned} \hat{M}^z(z, \bar{z}, \theta) = & dz + \mu_{\bar{z}}^z d\bar{z} + c^z + \theta^i \left(\alpha_{\bar{z}i}^{\frac{1}{2}} d\bar{z} + \gamma_i^{\frac{1}{2}} \right) + \frac{1}{2} \sum_{ij} \theta^i \theta^j \left(C_{\bar{z}ij}^0 d\bar{z} + c_{ij}^0 \right) \\ & + \sum_{p=3}^N \sum_{1 \leq i_1 \dots i_p \leq N} \frac{1}{p!} \theta^{i_1} \dots \theta^{i_p} \left(C_{\bar{z}i_1 \dots i_p}^{1-\frac{p}{2}} d\bar{z} + c_{i_1 \dots i_p}^{1-\frac{p}{2}} \right). \end{aligned} \quad (5)$$

The Beltrami differential $\mu_{\bar{z}}^z(z, \bar{z})$ and the classical fields $C_{\bar{z}i_1 \dots i_p}^{1-\frac{p}{2}}(z, \bar{z})$, $p = 1, \dots, N$, (with ghost number zero) define the conformally invariant left sector of the (N, N') 2D-supergravity. The anticommuting fields $\alpha_{\bar{z}i}^{\frac{1}{2}}(z, \bar{z})$ are identified as the holomorphic parts of the N gravitini (eventually they will be the sources of the N -supersymmetry currents) and the commuting fields $C_{\bar{z}ij}^0(z, \bar{z})$ are the components along \bar{z} of the commuting $O(N)$ gauge field which gauges the $O(N)$ rotations. The other classical fields contained in \hat{M}^z , $C_{\bar{z}i_1 \dots i_p}^{1-\frac{p}{2}}(z, \bar{z})$, $p = 2, \dots, N$, gauge the internal fermionic and bosonic symmetries of 2D-supergravity of rank N . Thus, the

holomorphic components of the 2D-supergravity multiplet are quite simply identified as the antisymmetric tensor representations of $O(N)$. The fields $c_{i_1 \dots i_p}^{1-\frac{p}{2}}(z, \bar{z})$ are their ghosts, with conformal weight $\frac{p}{2} - 1$. $C_{\bar{z}i_1 \dots i_p}^{1-\frac{p}{2}}(z, \bar{z})$ and $c_{i_1 \dots i_p}^{1-\frac{p}{2}}(z, \bar{z})$ have opposite statistics.

It is easy to verify that, as required by supersymmetry, the number of commuting fields in this multiplet equals the number of anticommuting fields. Indeed, the number of independent fields contained in the θ polynomials $\theta^{i_1} \dots \theta^{i_p} C_{\bar{z}i_1 \dots i_p}^{1-\frac{p}{2}}$ of rank p is equal to ${}_N C_p$. Since $0 = (1 - 1)^N$, one has $\sum_p {}_N C_{2p} = \sum_p {}_N C_{2p+1}$, which shows the required equality between the number of bosons and fermions, necessary to ensure the eventual closure of the 2D-supergravity algebra.

The BRST symmetry is defined by the following straightforward generalization of eq. (3):

$$\hat{d}\hat{M}^z = \hat{M}^z \partial_z \hat{M}^z - \frac{1}{4} \sum_{i=1}^N (D_i \hat{M}^z)^2, \quad (6)$$

$$\hat{M}^z = M_{\bar{z}}^z d\bar{z} + C^z. \quad (7)$$

One has the closure relation $\hat{d}^2 = 0$, that is $s^2 = 0$, if and only if

$$D_i D_j + D_j D_i = 2\delta_{ij} \partial_z. \quad (8)$$

Therefore, one has $D_i = \partial_{\theta^i} + \theta^i \partial_z$ in eq. (6).

The vanishing curvature condition (6) can also be understood as a realization of the abstract algebra (8). The BRST transformations of the classical and ghost superfields are then easily extracted from (6) as

$$sM_{\bar{z}}^z = \partial_{\bar{z}} C^z + C^z \partial_z M_{\bar{z}}^z - M_{\bar{z}}^z \partial_z C^z - \frac{1}{2} \sum_{i=1}^N D_i C^z D_i M_{\bar{z}}^z, \quad (9)$$

$$sC^z = C^z \partial_z C^z - \frac{1}{4} \sum_{i=1}^N (D_i C^z)^2. \quad (10)$$

Eq. (9) gives the classical gauge transformations of all components of the 2D-supergravity multiplet, simply by changing the ghosts into infinitesimal parameters with the opposite statistics.

Let us notice that one can deduce the full BRST algebra from the sole knowledge of the BRST transformation of the superfield C^z . Indeed, the complete BRST equations (6), can be directly obtained from (10) with the substitutions

$$s \rightarrow d + s, \quad C^z \rightarrow M^z + C^z. \quad (11)$$

It follows that the determination of the ghost transformation obtained from the OPE techniques in superconformal quantum field theory, as shown in ref. [2], would also indirectly permit the determination of the gauge fields associated to these ghosts and of their transformation laws.

3 The antighosts and the gauge-fixed action

Usually antighosts are directly introduced as conjugate variables to the ghosts. This is a consistent approach since one is mainly interested in superstring theory expressed in the (super)conformal gauge. In order to remain in a geometrical framework, it is however interesting to introduce in a gauge-independent way the conjugates of all fields contained in $\hat{M}^z(z, \bar{z}, \theta)$. To do so, we will use the BV formalism [4], where the (super)antifields are naturally the duals to the (super)fields [8]. The usual antighosts will be introduced afterwards via an appropriate choice of the gauge function.

Let us denote the antifields of $M_{\bar{z}}^z$ and C^z as ${}^*M_{zz}$ (with ghost number -1) and ${}^*C_{zz\bar{z}}$ (with ghost number -2), respectively, and define

$${}^*\hat{M}_z(z, \bar{z}, \theta) = {}^*M_{zz}dz + {}^*C_{zz\bar{z}}dzd\bar{z}. \quad (12)$$

The $O(N)$ superspace decomposition of ${}^*\hat{M}_z$ is

$$\begin{aligned} {}^*\hat{M}_z(z, \bar{z}, \theta) &= \frac{1}{N!} \epsilon_{i_1 \dots i_N} \theta^{i_1} \dots \theta^{i_N} ({}^*\mu_{zz}dz + {}^*c_{zz\bar{z}}dzd\bar{z}) \\ &+ \sum_{p=0}^{N-1} \frac{1}{p!} \epsilon_{i_1 \dots i_N} \theta^{i_1} \dots \theta^{i_p} ({}^*C_{zz i_{p+1} \dots i_N} dz + {}^*c_{zz\bar{z} i_{p+1} \dots i_N} dzd\bar{z}). \end{aligned} \quad (13)$$

The invariant BV action which determines the BRST symmetry (6) is the part with ghost number $g = 0$ of

$$I_{BV} = \int d^2z d^N\theta {}^*\hat{M}_z \hat{G}^z, \quad (14)$$

where

$$\hat{G}^z = d\hat{M}^z - \hat{M}^z \partial_z \hat{M}^z + \frac{1}{4} \sum_{i=1}^N (D_i \hat{M}^z)^2. \quad (15)$$

This BV action satisfies a (first rank) master BV equation. This means that it defines a nilpotent differential operator BRST, given by

$$s\hat{M}^z = \frac{\delta I_{BV}^{g=0}}{\delta {}^*\hat{M}_z}, \quad s {}^*\hat{M}_z = \frac{\delta I_{BV}^{g=0}}{\delta \hat{M}^z}. \quad (16)$$

The first equation is identical to the BRST transformation law found earlier for \hat{M}^z ; the second equation, which expresses the BRST transformation of antifields, implies eventually that the currents, in the superconformal gauge, are BRST-exact.

This form of the action (14) (prior to any kind of gauge fixing) indicates the rather deep connection of the theory with a topological BF type system. Notice that it contains no classical part, since the part with ghost number zero of \hat{G}^z in I_{BV} consists of ghosts only.

One naturally considers the gauge where one imposes that the gauge fields contained in the expansion of $\hat{M}^z(z, \bar{z}, \theta)$ are set equal to a background value, which we shall denote as $M_{\bar{z},bg}^z(z, \bar{z}, \theta)$. (The “superconformal gauge” is obtained for $M_{\bar{z},bg}^z = 0$.) This choice of gauge implies the introduction of antighosts, with the following superfield expansion

$$B_z(z, \bar{z}, \theta) = \frac{1}{N!} \epsilon_{i_1 \dots i_N} \theta^{i_1} \dots \theta^{i_N} b_{zz} dz + \frac{1}{(N-1)!} \epsilon_{i_1 i_2 \dots i_N} \theta^{i_2} \dots \theta^{i_N} \beta_{zz}^{i_1} dz + \dots, \quad (17)$$

and the following BV gauge function:

$$Z_{GF} = \int d^2 z d^N \theta B_z (M_{\bar{z}}^z - M_{\bar{z},bg}^z). \quad (18)$$

It implies

$$M_{\bar{z}}^z = M_{\bar{z},bg}^z, \quad {}^*M_{zz} = \frac{\delta Z_{GF}}{\delta M_{\bar{z}}^z} = B_z, \quad {}^*C_{zz\bar{z}} = \frac{\delta Z_{GF}}{\delta C^z} = 0. \quad (19)$$

The BV action then reduces to an action I_{GF} which only depends on the ghosts and antighosts

$$\begin{aligned} I_{GF} &= \int d^2 z d^N \theta B_z s M_{\bar{z}}^z |_{M_{\bar{z}}^z = M_{\bar{z},bg}^z} \\ &= \int d^2 z d^N \theta B_z \left(\partial_{\bar{z}} C^z + C^z \partial_z M_{\bar{z},bg}^z - M_{\bar{z},bg}^z \partial_z C^z - \frac{1}{2} \sum_{i=1}^N D_i C^z D_i M_{\bar{z},bg}^z \right). \end{aligned} \quad (20)$$

Let us give for completeness the expression of I_{GF} after integration upon the supercoordinates θ^i :

$$I_{GF} = \int d^2 z \left(b_{zz} s \mu_{\bar{z}}^z + \sum_{i_1 \dots i_p} \beta_{zz}^{i_1 \dots i_p} s M_{\bar{z}}^z |_{M_{\bar{z}}^z = M_{\bar{z},bg}^z} \right). \quad (21)$$

The detailed expressions of $s \mu_{\bar{z}}^z = \partial_{\bar{z}} C^z + \dots$ and $s M_{\bar{z}}^z |_{M_{\bar{z}}^z = M_{\bar{z},bg}^z} = \partial_{\bar{z}} m_{i_1 \dots i_p}^z + \dots$, $1 \leq p \leq N$, follow from the decomposition of eq. (9). We will shortly comment on the structure of these BRST transformations of the fields $M_{\bar{z}}^z |_{M_{\bar{z}}^z = M_{\bar{z},bg}^z}$ in component formalism.

4 The currents and the background symmetry

We can define the following supercurrent from (20):

$$\begin{aligned} J_z &= \frac{\delta I_{GF}}{\delta M_{\bar{z},bg}^z} \\ &= \left(\frac{N}{2} - 2 \right) B_z \partial_z C^z - \partial_z B_z C^z + \frac{(-)^{\epsilon_B}}{2} \sum_{i=1}^N D_i B_z D_i C^z, \end{aligned} \quad (22)$$

where ϵ_B is odd (even) integer for anticommuting (commuting) B_z .

The property that the above superfield currents (22) truly represent the linear SCA, and that the currents that are obtained by its decompositions onto the various $O(N)$ representations are (classically) conserved, is warranted by the existence of the background gauge symmetry for I_{GF} , which generalizes the background reparametrization invariance of the purely bosonic case. Let us now determine this symmetry, using the fact that we have at our disposal the background superfield M_{GF}^z .

It is most convenient to represent the infinitesimal $O(N)$ superfield parameter of this background gauge symmetry by an $O(N)$ superfield ghost of opposite statistics, denoted as $\Lambda^z(z, \bar{z}, \theta)$. Then one can *ghostify* the infinitesimal background gauge transformations under the form of an associated nilpotent generator s_Λ defined by the following equation:

$$\begin{aligned} (d + s + s_\Lambda)\tilde{M}^z &= \tilde{M}^z \partial_z \tilde{M}^z - \frac{1}{4} \sum_{i=1}^M (D_i \tilde{M}^z)^2, \\ \tilde{M}^z &= \hat{M}^z + \Lambda^z, \end{aligned} \quad (23)$$

where one assigns a new distinct ghost numbers to s_Λ and Λ with $s\Lambda^z = 0$. The differential operators s and s_Λ anticommute. One can find the transformation of the antighost B under s_Λ such that the action I_{GF} is invariant under the the background gauge symmetry, $s_\Lambda I_{GF} = 0$:

$$\begin{aligned} s_\Lambda M_{\bar{z},bg}^z &= \partial_{\bar{z}} \Lambda^z + \Lambda^z \partial_z M_{\bar{z},bg}^z - M_{\bar{z},bg}^z \partial_z \Lambda^z - \frac{1}{2} \sum_{i=1}^N D_i \Lambda^z D_i M_{\bar{z},bg}^z, \\ s_\Lambda C^z &= C^z \partial_z \Lambda^z + \Lambda^z \partial_z C^z - \frac{1}{2} \sum_{i=1}^N D_i \Lambda^z D_i C^z, \\ s_\Lambda B_z &= \Lambda^z \partial_z B_z + \left(2 - \frac{N}{2}\right) \partial_z \Lambda^z B_z - \frac{1}{2} \sum_{i=1}^N D_i \Lambda^z D_i B_z. \end{aligned} \quad (24)$$

This background gauge symmetry determines the Slavnov identity of the theory, and also ensures that the gauge-fixed action has the relevant superconformal symmetry of rank N .

5 Pure component formalism and the cases $N = 2, 3$ and 4

We now show that the sole knowledge of a SCA in component formalism would also provide the full information about the holomorphic Beltrami-like decomposition of 2D-supergravity and the BRST quantization of worldsheets with extended supersymmetry.

In full generality, a linear SCA is a super Lie algebra, whose structure coefficient determines a BRST symmetry of the following form

$$sc^z = c^z \partial_z c^z - \frac{1}{4} \gamma^i \gamma_i,$$

$$sm^a = c^z \partial_z m^a - w(m) m^a \partial_z c^z + f_{bc}^a m^b m^c + g_{bc}^a m^b \partial_z m^c, \quad (25)$$

where f_{bc}^a and g_{bc}^a are (constant) structure coefficients such that $s^2 = 0$ on all fields and $w(m)$ is the conformal weight of the ghost m^a . $s\gamma^i$ has an expression analogous to sm^a . It is convenient to separate the reparametrization ghost c^z and the local worldsheet supersymmetry ghosts γ^i , $1 \leq i \leq N$, from the rest of the ghosts denoted as m^a which are either odd or even and correspond to the internal gauge bosonic and fermionic symmetries of the SCA.

The holomorphic components of the classical gauge fields can now be defined directly. They are the Beltrami differential $\mu_{\bar{z}}^z$, in correspondence with c^z , the N gravitini $\alpha_{\bar{z}}^i$, in correspondence with γ^i , and the components of 1-forms along $d\bar{z}$, $M_{\bar{z}}^a d\bar{z}$, in correspondence with the ghosts m^a . One defines the unified objects

$$\begin{aligned} c^z \rightarrow \hat{\mu}^z &= dz + \mu_{\bar{z}}^z d\bar{z} + c^z, \\ \gamma^i \rightarrow \hat{\alpha}^i &= \alpha_{\bar{z}}^i d\bar{z} + \gamma^i, \\ m^a \rightarrow \hat{M}^a &= M_{\bar{z}}^a d\bar{z} + m^a. \end{aligned} \quad (26)$$

Then the full BRST algebra is simply read off from (25) as (for convenience we now incorporate the fields $\hat{\alpha}^i$ as a part of the \hat{M}^a)

$$\begin{aligned} (d+s)\hat{\mu}^z &= \hat{\mu}^z \partial_z \hat{\mu}^z - \frac{1}{4} \hat{\alpha}^a \hat{\alpha}_a, \\ (d+s)\hat{M}^a &= \hat{\mu}^z \partial_z \hat{M}^a - w(m) \hat{M}^a \partial_z \hat{\mu}^z + f_{bc}^a \hat{M}^b \hat{M}^c + g_{bc}^a \hat{M}^b \partial_z \hat{M}^c. \end{aligned} \quad (27)$$

By projection in ghost numbers and form degrees, the only nontrivial terms stemming from these equations give at ghost number 2 the s -transformations of the ghosts as in eq. (25), and at ghost number 1 the s -transformations of the gauge fields as

$$\begin{aligned} s\mu_{\bar{z}}^z &= \partial_{\bar{z}} c^z + c^z \partial_z \mu_{\bar{z}}^z - \mu_{\bar{z}}^z \partial_z c^z - \frac{1}{2} \alpha_{\bar{z}}^a \gamma_a, \\ sM_{\bar{z}}^a &= \mp \partial_{\bar{z}} m^a + c^z \partial_z M_{\bar{z}}^a + w(m) M_{\bar{z}}^a \partial_z c^z \\ &\quad \pm \mu_{\bar{z}}^z \partial_z m^a - w(m) m^a \partial_z \mu_{\bar{z}}^z \\ &\quad + f_{bc}^a (m^b M_{\bar{z}}^c \pm M_{\bar{z}}^b m^c) + g_{bc}^a (m^b \partial_z M_{\bar{z}}^c \pm M_{\bar{z}}^b \partial_z m^c). \end{aligned} \quad (28)$$

The rule for the sign in the above formula is that one has an upper sign when the ghost m^a on the right is commuting (corresponding to an anticommuting gauge field $M_{\bar{z}}^a$).

It is ensured by construction that the BRST transformation laws (28) are nilpotent. Moreover, the projection in the component formalism of the superfield equations (6) turn out to be

of the form of these last equations. The apparent complexity of the latter justifies the $O(N)$ superfield notation, which captures the whole information about the gauge symmetries under the form of a single zero curvature condition.

The BRST invariant action is as in eq. (21). The gauge fields in the action are the sources of the currents

$$T_{zz} = \frac{\delta I_{GF}}{\delta \mu_z^z}, \quad T_a = \frac{\delta I_{GF}}{\delta M_z^a}. \quad (29)$$

These currents would generate the BRST algebra (25) we started from by a mere application of the OPE technique. They are also the components of the supercurrent J_z defined in the previous section.

It is useful to illustrate these results explicitly in the cases $N = 2, 3$ and 4.

For $N = 2$, we have the Beltrami differential μ_z^z , two gravitini α_i , ($i = 1, 2$) and one commuting gauge field ρ . Their associated ghosts are denoted as c^z, γ_i and d , respectively. The transformation rules for the ghosts in $N = 2$ string are

$$\begin{aligned} sc^z &= c^z \partial_z c^z - \frac{1}{4} \gamma_i \gamma_i, \\ s\gamma_i &= c^z \partial_z \gamma_i - \frac{1}{2} \gamma_i \partial_z c^z - \frac{1}{2} \epsilon_{ij} \gamma_j d, \\ sd &= c^z \partial_z d + \frac{1}{2} \epsilon_{ij} \gamma_i \partial_z \gamma_j, \end{aligned} \quad (30)$$

where $\epsilon_{12} = -\epsilon_{21} = 1$ and repeated indices are summed.

For $N = 3$, the $O(3)$ decomposition gives the Beltrami differential μ_z^z , three gravitini α_i , three gauge fields ρ_i , and a fermionic field φ , ($i = 1, 2, 3$). Their associated ghosts are denoted as c^z, γ_i, c_i and δ , respectively. The transformation rules for the ghosts in $N = 3$ string are

$$\begin{aligned} sc^z &= c^z \partial_z c^z - \frac{1}{4} \gamma_i \gamma_i, \\ s\gamma_i &= c^z \partial_z \gamma_i - \frac{1}{2} \gamma_i \partial_z c^z - \frac{1}{2} \epsilon_{ijk} c_j \gamma_k, \\ sc_i &= c^z \partial_z c_i + \frac{1}{2} \epsilon_{ijk} \gamma_j \partial_z \gamma_k - \frac{1}{2} \gamma_i \delta + \frac{1}{4} \epsilon_{ijk} c_j c_k, \\ s\delta &= c^z \partial_z \delta + \frac{1}{2} \delta \partial_z c^z - \frac{1}{2} \gamma_i \partial_z c_i, \end{aligned} \quad (31)$$

where ϵ_{ijk} is the structure constant for $SU(2)$.

For $N = 4$, the $O(4)$ decomposition gives the Beltrami differential, four gravitini, six commuting fields that one can assemble into a gauge field for $O(4) \sim SU(2) \times SU(2)$ rotations, four anticommuting fields for an internal local supersymmetry and one commuting field for an internal local $U(1)$ symmetry. Associated with these, we have the reparametrization ghost c^z ,

four supersymmetry ghosts γ^a , six $SU(2)_{k+} \times SU(2)_{k-}$ symmetry ghosts c_{\pm}^i and one commuting δ^a and $U(1)$ ghost d [9, 6]. Here the double index notation $a = (\alpha, \bar{\alpha})$; $\alpha, \bar{\alpha} = 1, 2$ is used and $i = 1, 2, 3$ [10]. This corresponds to the so-called large $N = 4$ SCA.

The transformation rules for ghosts in $N = 4$ string can be written in concise form as:

$$\begin{aligned}
sc^z &= c^z \partial_z c^z - \frac{1}{4} \gamma_a \gamma^a, \\
s\gamma^a &= c^z \partial_z \gamma^a - \frac{1}{2} \gamma^a \partial_z c^z - R^{+,i}{}_b{}^a c_{+,i} \gamma^b - R^{-,i}{}_b{}^a c_{-,i} \gamma^b, \\
sc_{\pm}^i &= c^z \partial_z c_{\pm}^i - \frac{1}{2} \epsilon^i{}_{jk} c_{\pm}^j c_{\pm}^k - \frac{1}{2} (1 \mp x) R^{\pm,i}{}_{ab} \partial_z \gamma^a \gamma^b \pm R^{\pm,i}{}_{ab} \delta^a \gamma^b, \\
s\delta^a &= c^z \partial_z \delta^a + \frac{1}{2} \delta^a \partial_z c^z + \frac{1}{2} (1+x) R^{+,i}{}_b{}^a \gamma^b \partial_z c_{+,i} - \frac{1}{2} (1-x) R^{-,i}{}_b{}^a \gamma^b \partial_z c_{-,i} \\
&\quad - R^{+,i}{}_b{}^a \delta^b c_{+,i} - R^{-,i}{}_b{}^a \delta^b c_{-,i} - \frac{1}{2} \gamma^a \partial_z d, \\
sd &= c^z \partial_z d - \frac{1}{2} \delta_a \gamma^a.
\end{aligned} \tag{32}$$

The *free* parameter $x \equiv \frac{k^+ - k^-}{k^+ + k^-}$ measures the asymmetry between the two $SU(2)$ current algebras. Its occurrence in the BRST algebra in component formalism is a peculiarity of the case $N = 4$ and is allowed by the local isomorphism between $O(4)$ and $SU(2) \times SU(2)$ [9]. The $SU(2)$ representation matrices $R^{\pm,i}{}_a{}^b$ have the values

$$R^{+,i}{}_{(\alpha,\bar{\alpha})}{}^{(\beta,\bar{\beta})} = \begin{cases} \frac{1}{2} \bar{\sigma}^i{}_{\alpha}{}^{\beta} & \text{if } \bar{\alpha} = \bar{\beta} = 1 \\ \frac{1}{2} \sigma^i{}_{\alpha}{}^{\beta} & \text{if } \bar{\alpha} = \bar{\beta} = 2 \\ 0 & \text{otherwise} \end{cases}, \quad R^{-,i}{}_{(\alpha,\bar{\alpha})}{}^{(\beta,\bar{\beta})} = \begin{cases} \frac{1}{2} \bar{\sigma}^i{}_{\alpha}{}^{\bar{\beta}} & \text{if } \alpha = \beta = 1 \\ \frac{1}{2} \sigma^i{}_{\alpha}{}^{\bar{\beta}} & \text{if } \alpha = \beta = 2 \\ 0 & \text{otherwise} \end{cases}, \tag{33}$$

where $\sigma^i = (\sigma^3, \sigma^+, \sigma^-)$ are the Pauli matrices in the Cartan basis and $\bar{\sigma}^i = (\sigma^3, -\sigma^+, -\sigma^-)$. Indices are raised and lowered with the invariant tensors g^{ij}, η_{ab} and their inverse given by

$$g^{+-} = 2, \quad g^{33} = 1; \quad \eta_{(\alpha\bar{\alpha})(\beta\bar{\beta})} = \frac{1}{2} \bar{\eta}_{\alpha\beta} \bar{\eta}_{\bar{\alpha}\bar{\beta}}, \quad \text{with } \bar{\eta}_{12} = \bar{\eta}_{21} = 1, \quad \bar{\eta}_{11} = \bar{\eta}_{22} = 0. \tag{34}$$

The way the classical gauge fields associated to these ghosts transform is obtained by applying eq. (28), using the structure coefficients that one can read from eqs. (30)-(32).

It is straightforward to verify on these examples the general results of the last sections. In particular, the transformation rules for ghosts follow from eq. (10) in terms of $N = 2, 3$ and 4 superfields for each case,¹ and the preceding results are obtained just by projecting on each components.

¹For $N = 4$, the parameter x must be introduced in the ghost and antighost superfield decomposition.

6 The conformal anomaly

We now give a general formula for the value of the coefficient of conformal anomaly of the ghost system defined by the action I_{GF} . To compute the anomaly, one can put $M_{\bar{z},bg}^z = 0$. Then

$$I_{GF} = \int d^2z \left(b_{zz} \partial_{\bar{z}} c^z + \sum_{i_1 \dots i_p} \beta_{zi_1 \dots i_p} \partial_{\bar{z}} c_{\bar{z}}^{i_1 \dots i_p} \right). \quad (35)$$

Since a system of conformal fields (A, B) with Lagrangian $A \partial_{\bar{z}} B$ has a conformal anomaly equal to $\pm 2(6n^2 - 6n + 1)$ where n is the conformal weight of the field A and the sign $+$ ($-$) occurs if A and B commute (anticommute) [11], the value of the conformal anomaly associated to the action I_{GF} is

$$c(N) = \sum_{p=0}^N (-)^{p+1} {}_N C_p 2 \left(6w(p)^2 - 6w(p) + 1 \right), \quad (36)$$

where $w(p) = \frac{p}{2} - 1$.

One has $c(0) = -26, c(1) = -15, c(2) = -6$, which express the well-known fact that superstrings for $N = 0, 1$ and 2 have critical dimensions $D = 26, 10$ and 4 respectively, and, for all values of $N \geq 3$,

$$c(N) = 0, \quad N \geq 3. \quad (37)$$

This formula can be proved algebraically by using the defining relation $(1+z)^N = \sum_{p=0}^N {}_N C_p z^p$. In the next section we will demonstrate it in another way which emphasizes the string embedding property. Eq. (37) indicates that all superstrings based on the full $O(N)$ superspace have vanishing critical dimension for $N \geq 3$ [7, 10, 2], and can therefore only be of a purely topological nature.

Let us also consider the question of finding the anomaly, defined as the local 3-form solution modulo \hat{d} -exact terms of the consistency equation $\int \hat{d}\hat{\Delta}_3 = 0$. The conventional consistent anomaly, which is the possible local counterterm which can possibly break the conformal BRST Ward identity, is the component Δ_2^1 with ghost number 1 of $\hat{\Delta}_3$, defined modulo d - and s -exact terms.

In the case of $N = 0$ supersymmetry, one has

$$N = 0 : \hat{\Delta}_3 = \hat{\mu}^z \partial_z \hat{\mu}^z \partial_{\bar{z}}^2 \hat{\mu}^z. \quad (38)$$

This expression is in correspondence with the property that the violation of the conservation of the holomorphic component of the energy-momentum tensor is proportional to $\partial_z^3 \mu_{\bar{z}}^z$ [5].

To generalize this expression to higher values of rank N , we must search for the possible completion of $\hat{\mu}^z \partial_z \hat{\mu}^z \partial_z^2 \hat{\mu}^z$ by terms such that the whole expression is \hat{d} -closed but not \hat{d} -exact. This can be achieved by finding zero-forms Δ_0^3 with ghost number three which must be BRST-closed, and contain $c^z \partial_z c^z \partial_z^2 c^z$.

Quite interestingly, power counting requirements (Δ_0^3 must be made of local terms with the same canonical dimension as $c^z \partial_z c^z \partial_z^2 c^z$) imply that we only have *local* solutions, i.e, solutions which involve only positive powers of the derivative ∂_z for $N = 0, 1, 2, 3$. Indeed, for higher values of N , power counting forbids Δ_0^3 to depend on the ghosts whose weight is bigger or equal to one, and thus the consistency equation cannot be satisfied.

We find the following expressions for the consistent anomalies in components:

$$\begin{aligned}
N = 1 \quad : \quad \hat{\Delta}_3 &= \hat{\mu}^z \partial_z \hat{\mu}^z \partial_z^2 \hat{\mu}^z - \hat{\mu}^z (\partial_z \hat{\alpha}^z)^2 + \frac{1}{2} \partial_z \hat{\mu}^z \hat{\alpha}^z \partial_z \hat{\alpha}^z, \\
N = 2 \quad : \quad \hat{\Delta}_3 &= \hat{\mu}^z \partial_z \hat{\mu}^z \partial_z^2 \hat{\mu}^z - \hat{\mu}^z (\partial_z \hat{\alpha}_i^z)^2 + \frac{1}{2} \partial_z \hat{\mu}^z \hat{\alpha}_i^z \partial_z \hat{\alpha}_i^z + \hat{\mu}^z \hat{\rho}^z \partial_z \hat{\rho}^z - \frac{1}{2} \epsilon_{ij} \hat{\rho}^z \hat{\alpha}_i^z \partial_z \hat{\alpha}_j^z, \\
&\quad (i, j = 1, 2), \\
N = 3 \quad : \quad \hat{\Delta}_3 &= \hat{\mu}^z \partial_z \hat{\mu}^z \partial_z^2 \hat{\mu}^z - \hat{\mu}^z (\partial_z \hat{\alpha}_i^z)^2 + \frac{1}{2} \partial_z \hat{\mu}^z \hat{\alpha}_i^z \partial_z \hat{\alpha}_i^z + \hat{\mu}^z \hat{\rho}_i^z \partial_z \hat{\rho}_i^z - \hat{\mu}^z (\hat{\varphi}^z)^2 \\
&\quad - \frac{1}{2} \hat{\rho}_1^z \hat{\rho}_2^z \hat{\rho}_3^z - \frac{1}{2} \epsilon_{ijk} \hat{\rho}_i^z \hat{\alpha}_j^z \partial_z \hat{\alpha}_k^z + \frac{1}{2} \hat{\rho}_i^z \hat{\alpha}_i^z \hat{\varphi}^z, \quad (i, j, k = 1, 2, 3),
\end{aligned} \tag{39}$$

where $\hat{\alpha}, \hat{\rho}, \hat{\varphi}$ are the unified fields as defined in eq. (26) for gravitini, gauge and fermionic fields.

The $N=4$ case is subtle because of the existence of a generator of dimension zero. For all other values of $N > 4$, there is no consistent anomaly. This is compatible with the fact that the sums of all ghost anomaly contributions vanish as shown in our computation in the first part of this section and also with the absence of central extension for $N > 4$ [7]. This property probably expresses that the conformal field theory is free of infinities for $N > 4$.

7 The superstring embedding

The ghost system of the $O(N)$ superspace is made of a tower of ghosts which describe the antisymmetric tensors of $O(N)$, with ${}_N C_p$ ghosts with weight $w(p)$ and statistics $(-1)^{p+1}$. We will show the interconnection of these systems when N varies.

Let us define the following notation which defines the N -superstring from the knowledge of its ghost spectrum:

$$N\text{-theory} \equiv \{ ({}_N C_p, w(p)) ; 0 \leq p \leq N \}, \tag{40}$$

where the first entry indicates degeneracy and the second the conformal weight. The antighosts

are implicitly defined as the duals of the ghosts, and the action is as in (21). The theory of rank $N - 1$ is thus represented by

$$(N - 1)\text{-theory} \equiv \{ ({}_{N-1}C_p, w(p)) ; 0 \leq p \leq N - 1 \}. \quad (41)$$

Thanks to this notation, it becomes almost obvious to see that embedding of the $(N - 1)$ -theory into the N -theory just follows from the relation

$${}_NC_p = {}_{N-1}C_p + {}_{N-1}C_{p-1}, \quad (42)$$

valid for all values of $N \geq 1$ and $p \geq 1$.

Indeed, this relation suggests considering the ghost system obtained by isolating in the N -theory one ghost with weight $w(1)$, ${}_{N-1}C_1$ ghosts with weight $w(2)$, \dots , ${}_{N-1}C_{p-1}$ ghosts with weight $w(p)$, and so on down to ${}_{N-1}C_{N-1} = 1$ ghost with weight $w(N)$. We can denote this subsystem as

$$\Delta N\text{-theory} \equiv \{ ({}_{N-1}C_{p-1}, w(p)) ; 1 \leq p \leq N - 1 \}. \quad (43)$$

But then, one has obviously from eq. (42)

$$N\text{-theory} \equiv (N - 1)\text{-theory} \cup \Delta N\text{-theory}. \quad (44)$$

For $N = 1, 2, 3$, the ΔN -theory are anomalous. More precisely: The $\Delta 1$ -theory is made from one (β, γ) ghost-antighost pair and has $c = 11$; the $\Delta 2$ -theory is made from one (β, γ) pair and one Grassmann-odd ghost-antighost pair contributing to the anomaly by the amount -2 ; this theory has thus $c = 11 - 2 = 9$; finally, the $\Delta 3$ -theory is made from one (β, γ) pair, two odd pairs contributing to the anomaly by the amount $c = -2 \times 2$ and one even pair contributing to the anomaly by the amount $c = -1$; the $\Delta 3$ -theory has thus $c = 11 - 4 - 1 = 6$.

For $N \geq 4$, one encounters a new regime. The $\Delta 4$ -theory is made from one (β, γ) pair, three odd pairs contributing to the anomaly by the amount $c = -2 \times 3$, three even pairs contributing to the anomaly by the amount $c = -1 \times 3$ and one odd pair contributing to the anomaly by the amount $c = -2$; the $\Delta 4$ -theory has thus $c = 11 - 6 - 3 - 2 = 0$. Since, on the other hand, we can easily verify that for the 3-theory $c = 0$, we see that the 4-theory is also anomaly free, which could of course be verified directly. Moreover, when computing the path integral over the ghost and antighost fields, since both the $\Delta 4$ -theory and 3-theory are separately anomaly free, the fields of the $\Delta 4$ -theory can be integrated out, and we obtain the result that the 3-theory is embedded in the 4-theory.

This can be pursued by induction. One can prove in this way (i) that the N -theory as well as the $\Delta(N+1)$ -theory are anomaly free for $N \geq 3$ and (ii), as a corollary, that the N -theory is embedded in the $(N+1)$ -theory.

8 Connection to purely topological actions

Let us conclude by noting that the (N, N') theories can be also viewed as topological 2D-gravity coupled to topological sigma models with bosonic and fermionic coordinates. The field spectrum of the left sector of a worldsheet with (N, N') local supersymmetry is made of 2^N classical gauge fields (including the Beltrami differential) that we denoted as $C_{i_1 \dots i_p}^{1-p/2}$, $1 \leq p \leq N$, and of 2^N ghosts, with an equal partition between bosons and fermions. One has a similar situation in the right sector, by replacing N by N' . For $N = 1$, it has been shown in refs. [3, 12] that the $(1, 1)$ 2D-supergravity BRST algebra can be twisted into that of pure topological 2D-gravity by field redefinitions mixing the gauge fields and the ghosts, and concluded that the $N = 1$ superstring can be viewed as topological 2D-gravity coupled to a topological sigma model. Quite remarkably this observation can be extended to the case of any given (N, N') local supersymmetry. Indeed, in the left sector, one can do field redefinitions involving the 2^N classical gauge fields and their ghosts to obtain 2^{N-1} sets of topological pairs Y^i, F^i , $1 \leq i \leq 2^{N-1}$, such that $sY^i = F^i$, $sF^i = 0$. For $N \geq 2$, the 2^{N-1} fields Y^i consists of the Beltrami differential, $2^{N-2} - 1$ bosons and 2^{N-2} fermions which can thus be interpreted as the coordinates of a topological sigma model. (For $N = 1$, only the Beltrami differential and its topological ghost occur.)

These field redefinitions, which modify the ghost numbers as well as the conformal weights, are such that the bosonic and fermionic ghosts F^i are quadratic products of some of the fields $C_{i_1 \dots i_p}^{1-p/2}$ by some of the ghosts $c_{i_1 \dots i_q}^{1-p/2}$. They can be read off from the general formula for the BRST transformation of (N, N') local supersymmetry. This is a mere generalization of getting the topological ghost $\Psi_{\bar{z}}^z$ in topological 2D-gravity as the product of the gravitino $\alpha_{\bar{z}}^{\frac{1}{2}}$ by its commuting ghost $\gamma^{\frac{1}{2}}$, $\Psi_{\bar{z}}^z = \alpha_{\bar{z}}^{\frac{1}{2}} \gamma^{\frac{1}{2}}$.

One can complete this picture by finding a ghost of ghost phenomenon which also generalizes that of the case $N = 1$, where the ghost of ghost for reparametrization is $\Phi^z = \gamma^{\frac{1}{2}} \gamma^{\frac{1}{2}}$. This ghost of ghost phenomenon simply takes into account all internal symmetries of $O(N)$ superspace.

It is then possible to redefine the antighosts as in refs. [3, 12], in a way which is the conjugate to the redefinitions of the ghosts. This amounts altogether to canonical transformations. In this way, we finally arrive at the conclusion that the BRST invariant action for the pure worldsheet

theories with (N, N') local supersymmetry can be also considered as topological 2D-gravity coupled to a topological sigma model with $2^{N-2} - 1$ bosonic coordinates and 2^{N-2} fermionic coordinates in the left sector, and $2^{N'-2} - 1$ bosonic coordinates and $2^{N'-2}$ fermionic coordinates in the right sector.

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References

- [1] N. Berkovits and C. Vafa, Mod. Phys. Lett. **A9** (1994) 653.
- [2] F. Bastianelli, N. Ohta and J. L. Petersen, Phys. Rev. Lett. **73** (1994) 1199; Phys. Lett. **B327** (1994) 35; N. Berkovits and N. Ohta, Phys. Lett. **B334** (1994) 72; N. Ohta and T. Shimizu, Phys. Lett. **B355** (1995) 127.
- [3] L. Baulieu, M. Green and E. Rabinovici, Phys. Lett. **B386** (1996) 91, and in preparation.
- [4] I. A. Batalin and G. A. Vilkovisky, Phys. Rev. **D28** (1983) 2567.
- [5] L. Baulieu and M. Bellon, Phys. Lett. **B196** (1987) 142; L. Baulieu, M. Bellon and R. Grimm, Phys. Lett. **B198** (1987) 343; L. Baulieu, R. Stora, Lectures given at July 1987 NATO ASI, *on Non-Perturbative Quantum Field Theory*, published in Proceedings of the Cargese Summer Institute, 1987 (QC174.45:N2:1987), Plenum Press.
- [6] M. Ademollo *et al.*, Phys. Lett. **62B** (1976) 105; Nucl. Phys. **B114** (1976) 297.
- [7] K. Schouten, Nucl. Phys. **B295** (1988) 634.
- [8] L. Baulieu, preprint hep-th/ 9512026, hep-th/9512027, to appear in Nucl. Phys. **B**.
- [9] A. Sevrin, W. Troost and A. Van Proeyen, Phys. Lett. **B208** (1988) 447; E. Ivanov, S. Krivonos and V. Leviant, Phys. Lett. **B215** (1988) 689.
- [10] F. Bastianelli and N. Ohta, Phys. Rev. **D50** (1994) 4051.
- [11] D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. **B271** (1986) 93.
- [12] L. Baulieu, Phys. Lett. **B288** (1992) 59.